Example

$$L = \lim_{x \to \infty} \left(\frac{2x+3x^2}{3x^2-1} \right)^{x^2-x}$$

Solution

$$\ln L = \ln\left(\lim_{x\to\infty} \text{''} \right)$$

$$= \lim_{x\to\infty} \ln\left(\left(\frac{2x+3x^2}{3x^2-1} \right)^{x^2-x}\right)$$

$$\ln L = \lim_{x\to\infty} \underbrace{(x^2-x)}_{\to\infty} \underbrace{\ln\left(\underbrace{\frac{2x+3x^2}{3x^2-1}}_{\to 1} \right)}_{\to 0}$$

$$= \lim_{x\to\infty} \frac{\overbrace{\ln(2x+3x^2) - \ln(3x^2-1)}^{\to 0}}{\underbrace{\frac{1}{x^2-x}}_{\to 0}}$$

$\frac{0}{0}$ form

l'H

$$= \lim_{x\to\infty} \frac{\frac{1}{2x+3x^2}(2+6x) - \frac{1}{3x^2-1}(6x)}{-(x^2-x)^{-2}\cdot(2x-1)}$$

$18x^3 - 18x^3 + 6x^2 - 12x^2 + \text{lower}$

$$= \lim_{x\to\infty} \left[\frac{(2+6x)(3x^2-1) - 6x(2x+3x^2)}{(2x+3x^2)(3x^2-1)} \right] \frac{(x^2-x)^2}{-(2x-1)}$$

$$= \lim_{x\to\infty} \left[\frac{-6x^2 + \text{lower}}{(3x^2+\text{lower})(3x^2+\text{lower})} \right] \frac{(x^2+\text{lower})^2}{(-2x+\text{lower})}$$

$$= \lim_{x\to\infty} \frac{x^6\left[-6 + \overset{\to 0}{\text{low}}\right]\left[\left((1+\overset{\to 0}{\text{lo}}\right)^2\right]}{x^5\left[(3+\text{lo})(3+\text{lo})\right]\cdot\left[-2+\text{lo}\right]}$$

$$= \lim_{x\to\infty} \frac{\cancel{-6}^{\,1}\cancel{x^6}\,x}{\cancel{-18}_{\,3}\cancel{x^5}_{\,1}} = +\infty.$$

$\therefore \ln L = \infty$

$L = e^{\infty} = \boxed{\infty}$.

Showing integrals converge (= finite #) or diverge (limit DNE).

(without actually doing the integral.)

Important facts.

$\int_1^\infty \frac{1}{x^p}\,dx \leftarrow$ converges if $p > 1$, diverges otherwise

$\int_0^1 \frac{1}{x^p}\,dx \leftarrow$ converges if $p < 1$, diverges otherwise.

$\boxed{\text{Example}}$ $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2+4}}\,dx$

Observe for $x \geq 1$

$$0 \leq \frac{1}{\sqrt{5}\,x} = \frac{1}{\sqrt{x^2+4x^2}} \leq \frac{1}{\sqrt{x^2+4}} \leq \frac{1}{\sqrt{x^2+0}} = \frac{1}{x}$$

$$\Rightarrow \underbrace{\int_1^\infty \frac{1}{\sqrt{5}\,x}\,dx}_{\frac{1}{\sqrt{5}}\int_1^\infty \frac{1}{x}\,dx = \infty} \leq \int_1^\infty \frac{1}{\sqrt{x^2+4}}\,dx \leq \int_1^\infty \frac{1}{x}\,dx$$

$\Downarrow$ Thus $= \boxed{\infty}$.

$\boxed{\text{Ex}}$ $\displaystyle\int_0^1 \frac{3+\cos(7x^2)}{\sqrt{x}+2x^2}\,dx = ?$

does this converge?

Solution for $0<x<1$,

$$\frac{4}{\sqrt{x}} \geq \frac{3+\cos(7x^2)}{\sqrt{x}+2x^2} \geq \frac{2}{\sqrt{x}+2\sqrt{x}} = \frac{2}{3\sqrt{x}}$$

$$\Rightarrow \int_0^1 \frac{4}{\sqrt{x}}\,dx \le \int_0^1 \frac{3+\cos(7x^2)}{\sqrt{x}+2x^2}\,dx \le \int_0^1 \frac{2}{3\sqrt{x}}\,dx$$

bigger

$$8 = 4\int_0^1 \frac{1}{x^{1/2}}\,dx \qquad \frac{2}{3}\cdot 2 = \frac{4}{3}$$

converges

$$\int_0^1 x^{-1/2}\,dx = \lim_{a\to 0^+}\int_a^1 x^{-1/2}\,dx = \lim_{a\to 0^+} 2x^{1/2}\Big|_a^1$$

$$= \lim_{a\to 0^+}\left(2-2\underbrace{a^{1/2}}_{\to 0}\right) = 2$$

$$\therefore\quad 8 \ge \int_0^1 \frac{3+\cos(7x^2)}{\sqrt{x}+2x^2}\,dx \ge \frac{4}{3}$$

Converges!

$\boxed{\text{Ex}}$ Does $\displaystyle\int_0^\infty \frac{x+e^{-x}}{2^x+4}\,dx$ converge?

Solution. $0 < x < \infty$

$$\frac{x}{2^x+4\cdot 2^x} \le \frac{x+e^{-x}}{2^x+4} \le \frac{x+1}{2^x}$$

$$\frac{x}{2^x+4\cdot 2^x} = \frac{x}{5\cdot 2^x} = \frac{1}{5}x\cdot 2^{-x} \qquad \frac{x+1}{2^x} = (x+1)2^{-x}$$

$$\Rightarrow \quad \frac{1}{5}\int_0^\infty x2^{-x}\,dx \le \int_0^\infty \frac{x+e^{-x}}{2^x+4} \le \int_0^\infty (x+1)2^{-x}\,dx$$

$$\int_0^\infty x2^{-x}\,dx = \lim_{b\to\infty}\left[\int_0^b x2^{-x}\,dx\right]$$

parts

$u=x \quad du=dx$

$dv=2^{-x} \quad v=\frac{-1}{\ln 2}2^{-x}$

$$=\lim_{b\to\infty}\left[\frac{-x2^{-x}}{\ln 2}+\frac{1}{\ln 2}\underbrace{\int 2^{-x}\,dx}_{-\frac{2^{-x}}{\ln 2}}\right]_0^b$$

$$=\lim_{b\to\infty}\left[\frac{-x2^{-x}}{(\ln 2)}-\frac{2^{-x}}{(\ln 2)^2}\Big|_0^b\right]$$

$$=\lim_{b\to\infty}\left[\underbrace{\frac{-b2^{-b}}{\ln(2)}}_{\to 0}-\underbrace{\frac{2^{-b}}{(\ln 2)^2}}_{\to 0}-\left(0-\frac{1}{(\ln 2)^2}\right)\right]$$

$$\lim_{b\to\infty} b2^{-b} = \lim_{b\to\infty}\frac{b}{2^b} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{b\to\infty}\frac{1}{(\ln 2)2^b}=0$$

$$=\frac{1}{(\ln 2)^2} \quad " \int_0^\infty x2^{-x}\,dx"$$

Similarly $\displaystyle\int_0^\infty 2^{-x}\,dx = \lim_{b\to\infty}\frac{-1}{\ln 2}2^{-x}\Big|_0^b$

$$=\lim_{b\to\infty}\left(\underbrace{\frac{-1}{\ln 2}2^{-b}}_{\to 0}-\left(\frac{-1}{\ln 2}\right)\right)=\frac{1}{\ln 2},$$

$$\frac{1}{5}\int_0^\infty x\,2^{-x}\,dx \le \int_0^\infty \frac{x+e^{-x}}{2^x+4} \le \int_0^\infty (x+1)2^{-x}\,dx$$

$$\Rightarrow \frac{1}{5}\cdot\frac{1}{(\ln 2)^2} \le \int_0^\infty \frac{x+e^{-x}}{2^x+4}\,dx \le \frac{1}{(\ln 2)} + \frac{1}{(\ln 2)^2}$$

$\Rightarrow$ converges !

What have we been doing:

If $\quad 0 \le f(x) \le g(x)$

And if $\int_a^b g(x)\,dx$ converges, then

$\int_a^b f(x)\,dx$ converges and

$$0 \le \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

On the other hand, if $\int_a^b f(x)\,dx$ diverges $(=\infty)$,

then $\quad \int_a^b g(x)\,dx = \infty$ also.

Bounding fractions by simpler fractions

☆ Locate the dominant term in numerator & denom.

Eg: $$\int_5^\infty \frac{e^x + x^{1000} - 2\cos(x)}{x^2 - 2x + 75}\,dx$$

dominant

$$\frac{\lesssim e^x}{\lesssim x^2} \leq \frac{e^x + x^{1000} - 2\cos(x)}{x^2 - 2x + 75} \leq \frac{\lesssim e^x}{\lesssim x^2}$$

Eg: $$\int_0^1 \frac{e^x(\sqrt{x}+5)}{x^4 - 2\sqrt{x} + 1}\,dx$$